New Mathematics from Lambda Calculus

Antonino Salibra
Università Ca'Foscari, Venezia

In honor of Simone Martini

Church Algebras

Definition

An algebra $\mathbf{A}$ is a *Church algebra* if it admits a ternary operator $q(x, y, z)$ and two constants $0, 1$ such that, for all $x, y \in \mathbf{A}$:

$$q(1, x, y) = x \qquad q(0, x, y) = y$$

Examples :

- Rings : $q(x, y, z) = xy + (1 - x)z$.
- Boolean algebras : $q(x, y, z) = (x \wedge y) \vee (\neg x \wedge z)$.
- Lambda models : $1 = [\![\lambda xy.x]\!]; \quad 0 = [\![\lambda xy.y]\!]; \quad q(x, y, z) = [\![(xy)z]\!]$.
- Combinatory algebras : $1 = \mathbf{k}; \quad 0 = \mathbf{sk}; \quad q(x, y, z) = (xy)z$.

A digression : semantic incompleteness in the λ-calculus

The problem

Given a class of models of the pure λ-calculus, is it the case that *every* λ-theory is representable in it ?

Some (ad hoc) incompleteness results :

- Scott semantics (Honsell and Ronchi della Rocca 1992),
- Stable semantics (Gouy 1995),
- Strongly stable semantics (Salibra 2001).

A *general* incompleteness result :

Theorem (Manzonetto and Salibra 2006)

The λ-models arising from cpo-based semantics are *indecomposable*, and there exist λ-theories admitting only *decomposable* models.

Factor congruences and decomposition

Decomposition

$$\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$$
$$\Updownarrow$$
$$\exists \theta, \overline{\theta} \text{ such that } \mathbf{B} \cong \mathbf{A}/\theta \text{ and } \mathbf{C} \cong \mathbf{A}/\overline{\theta}$$
$$\Updownarrow$$
$$\exists \theta, \overline{\theta} \text{ as above, and moreover } \theta \cap \overline{\theta} = \Delta \text{ and } \theta \circ \overline{\theta} = \nabla$$

$(\theta, \overline{\theta})$ is called *a pair of complementary factor congruences* (CFC-pair).

An algebra **A** is

- *indecomposable* if it does not admit any non trivial CFC-pair.
- *simple* if $Con(\mathbf{A}) = \{\Delta, \nabla\}$.

Central elements

Definition

An element c of a double-pointed algebra is *central* if $(\theta(1,c), \theta(0,c))$ is a CFC-pair.

Within Church Algebras

$$\text{The BA of central elements} \quad \rightleftarrows \quad \text{The BA of CFC-pairs}$$

$$\begin{array}{rcl} c \in \mathrm{Ce}(\mathbf{A}) & \mapsto & (\theta(1,c), \theta(0,c)) \\ (\theta, \bar{\theta}) & \mapsto & \text{The unique } c \text{ such that } 1\theta c \bar{\theta} 0 \end{array}$$

BA-operations on central elements

$$x \wedge y = q(x, y, 0); \quad x \vee y = q(x, 1, y); \quad \neg x = q(x, 0, 1)$$

Equational characterization

$c \in \mathbf{A}$ is central iff

1. $q(c, 1, 0) = c$
2. $q(c, x, x) = x$
3. $q(c, q(x_1, x_2, x_3), q(y_1, y_2, y_3)) = q(q(c, x_1, y_1), q(c, x_2, y_{2,}), q(c, x_3, y_3))$

Non-trivial central elements in λ-models

The λ-term $\Omega = (\lambda x.xx)(\lambda x.xx)$ is *easy* : it can be consistently equated to any other closed λ-term.

Fact

In the term model of the λ-theory $\theta(1,\Omega) \cap \theta(0,\Omega)$, where $1 = \lambda xy.x, 0 = \lambda xy.y$, Ω is a *non-trivial* central element (hence the term model of that λ-theory is decomposable).

Lemma

Indecomposable lambda-algebras are closed by subalgebras.

Corollary

Any model of the λ-theory $\theta(1,\Omega) \cap \theta(0,\Omega)$ is decomposable (since if $\mathcal{M}$ is a model of the λ-calculus then the term model of $\mathrm{Th}(\mathcal{M})$ is a subalgebra of $\mathcal{M}$).

Incompleteness

Scott λ-algebras are simple, hence indecomposable

Proof :

- Let $\phi \neq \Delta$ be a congruence on a Scott λ-algebra $\mathbf{A}$. Suppose $a\phi b$ with $a \not\leq b$.
- Given $c \in \mathbf{A}$, $g_c(x) = \begin{cases} c & \text{if } x \not\leq b \\ \bot & \text{otherwise} \end{cases}$ is continuous.
- Let $d \in \mathbf{A}$ representing g_c (for all x, $dx = g_c(x)$).
- Hence $c = da \; \phi \; db = \bot$, and $\phi = \nabla$ by the arbitrariness of c.

Theorem

The Scott semantics of the λ-calculus is incomplete.

(no Scott model may have $\theta(1, \Omega) \cap \theta(0, \Omega)$ as theory)
More generally :

Theorem

The class of all indecomposable λ-models (including Scott, stable, strongly stable semantics) is incomplete.

Incompleteness 2

Let $(G, \to)$ be a directed graph. We define :

$$x \leftrightarrow_1 y \text{ if either } x \to y \text{ or } y \to x; \qquad x \leftrightarrow_{k+1} y \text{ if } \exists z\ x \leftrightarrow_k z \leftrightarrow_1 y$$

Theorem

The class of all partially ordered λ-models with bottom is incomplete.

Proof : Let $(G, \to)$ be a directed graph with a binary operation $\bullet$ on nodes and with a designated node 0 such that

1. $x \to y \ \Rightarrow\ z \bullet x \to z \bullet y$ and $x \bullet z \to y \bullet z$
2. $x \bullet x = 0$
3. $x \to y$ and $y \to x \ \Rightarrow\ x = y$

We define : $x \bullet_1 y = x \bullet y; \qquad x \bullet_{n+1} y = 0 \bullet (x \bullet_n y)$.

1. $x \to y \ \Rightarrow\ z \bullet_n x \to z \bullet_n y$ and $x \bullet_n z \to y \bullet_n z$
2. $x \bullet_n x = 0$

(a) $x \to y \ \Rightarrow_{(1)}\ 0 = x \bullet x \to x \bullet y \to y \bullet y = 0 \ \Rightarrow_{(3)}\ x \bullet y = 0$ and $y \bullet x = 0$

(b) $x \leftrightarrow_n z \leftrightarrow_1 y$ implies $0 =_{(\mathrm{Ind})} x \bullet_n z \leftrightarrow_1 x \bullet_n y$ implies $0 =_{(a)} 0 \bullet (x \bullet_n y) = x \bullet_{n+1} y$

(c) $\forall n. x \bullet_n y \neq 0$ implies that there is no path of $\leftrightarrow_1$ from x to y.

(d) The lambda theory axiomatised by $0xx = 0$, where $0 \equiv \Omega$ is the looping term and $x \bullet y = 0xy$, does not admit po-models with bottom.

Easy sets of lambda terms

Let **A** be a Church algebra. A finite subset $X \subseteq_{\text{fin}} A$ is *an easy set* if

$$\forall (b \subseteq X)\ \theta(b, 1) \vee \theta(X \setminus b, 0) \text{ is a consistent congruence}$$

Theorem (Manzonetto-Salibra)

Let **A** be a Church algebra and X be an easy set. Then there exists a congruence ϕ such that the principal filter $\phi\uparrow$ (of the congruences $\psi \supseteq \phi$) is isomorphic to the free Boolean algebra with X generators.

Corollary

The lattice of lambda theories contains (at the top) Boolean lattices of any finite cardinality.

Church algebras of dimension n

Definition

An algebra **A** is a *Church algebra of dimension n* (n-CA) if it admits a $(n+1)$-ary operator $q(x, y_1, \dots, y_n)$ and n constants $e_1, \dots, e_n$ such that for all $1 \le i \le n$

$$q(e_i, y_1, \dots y_n) = y_i$$

Examples :

- Lambda models :
  - $e_i = [\![\lambda x_1 \dots x_n . x_i]\!]$
  - $[\![q(x, y_1, \dots, y_n) = x y_1 \dots y_n]\!]$.
- n-sets on a set U : $\{(X_1, \dots, X_n) : X_i \subseteq U\}$;
  - $e_i = (\underbrace{\emptyset, \dots, \emptyset}_{(i-1)\text{ times}}, U, \emptyset, \dots, \emptyset)$,
  - $q(X, Y^1, \dots, Y^n) = (\bigcup_{j=1}^n X_j \cap Y^j_1, \dots, \bigcup_{j=1}^n X_j \cap Y^j_n)$.

n-central elements

If **A** is an n-CA, then $c \in A$ is called *n-central* if the sequence $(\theta(c,e_1),\dots,\theta(c,e_n))$ is a tuple of complementary factor congruences of **A**, meaning that :

- $\bigcap_{i\leq n}\theta(c,e_i) = \Delta$;
- for all $a_1,\dots,a_n \in A$, $q(c,a_1,\dots,a_n)$ is the unique element such that $a_i\ \theta(c,e_i)\ q(c,a_1,\dots,a_n)$ for all $1\leq i\leq n$;

.

If c is n-central, then $\mathbf{A} \cong \mathbf{A}/\theta(c,e_1) \times \dots \times \mathbf{A}/\theta(c,e_n)$.

Equational characterisation

$c \in \mathbf{A}$ is n-central iff :

1 : $q(c,e_1,\dots,e_n) = c$.

2 : $q(c,x,x,\dots,x) = x$ for every $x \in A$.

3 :

$$q(c,q(x_1^1,\dots,x_{n+1}^1),\dots,q(x_1^n,\dots,x_{n+1}^n))$$
$$= q(q(c,x_1^1,\dots,x_1^n),\dots,q(c,x_{n+1}^1,\dots,x_{n+1}^n)).$$

Proposition

The set of n-central elements of a n-CA $\mathbf{A}$ is a subalgebra of the reduct of $\mathbf{A}$ (i.e. : if $x, y_1, \ldots, y_n$ are n-central, then $q(x, y_1, \ldots, y_n)$ is n-central.)

Definition

A *Boolean algebra of dimension n* (n-BA) is a n-CA whose elements are all n-central.

Examples :

- The n-BA $(\{e_1, \ldots, e_n\}, q, e_1, \ldots, e_n)$ of *generalized truth values*.
- The n-CA of n-partitions of a set U :
  $\{(X_1, \ldots, X_n) : X_i \subseteq U, \ \bigcup_i X_i = U, \ X_i \cap X_j = \emptyset\}$;

A representation theorem

Any n-BA is isomorphic to a field of n-partitions.

The 2-elements Boolean algebra is *primal* : all Boolean functions are definable by AND, OR, NOT. This property is inherited by n-BAs.

Primality in n-BAs

The n-BA $(\{e_1, \dots, e_n\}, q, e_1, \dots, e_n)$ is primal and generates the variety of n-BAs.

Conclusion

Happy Birthday Simone !