Proofs and surfaces

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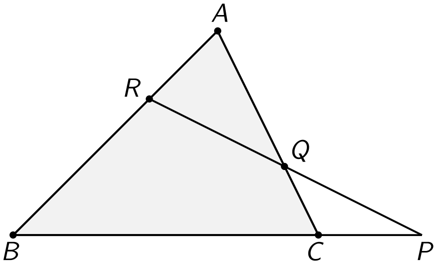
Incidence theorems

Incidence theorems in Euclidean or projective geometry state that some incidences follow from other incidences, where an incidence is a pair of a line and a point, together with the information whether the point lies on the line or not.

The Menelaus theorem



Pick a point P on the line BC, and so for Q and R

Characterise the alignment of P, Q, R

Menelaus configurations

For three points in the Euclidean plane $\mathbb{R}^2$,let

$$(X, Y; Z) =_{df} \begin{cases} \frac{XZ}{YZ}, & \text{if } Z \text{ is between } X \text{ and } Y, \\ -\frac{XZ}{YZ}, & \text{otherwise.} \end{cases}$$

if X, Y and Z are mutually distinct and colinear. Otherwise, we set $(X, Y; Z)$ to be undefined.
A sextuple (A, B, C, P, Q, R) ($ABCPQR$ for short) of points in $\mathbb{R}^2$ *is a Menelaus configuration* when

$$(B, C; P),\ (C, A; Q) \text{ and } (A, B; R) \text{ are defined}$$

and their product is -1.

Theorem. For a triangle ABC (with A, B, C not colinear), and points P, Q and R (different from the vertices) on the lines BC, CA and AB respectively, it holds that

$$P, Q, R \textit{ are colinear} \quad \textit{iff} \quad (B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1.$$

This theorem is an easy consequence of Thales' theorem.

Positive and negative colinearities

Given a sextuple $S = ABCPQR$, we set

$$
\begin{array}{ll}
col(S) = col^+(S) \cup \{col^-(S)\} & \text{colinearities} \\
col^+(S) = \{ABR, BCP, CAQ, PQR\} & \text{positive colinearities} \\
col^-(S) = \neg ABC & \text{negative colinearity.}
\end{array}
$$

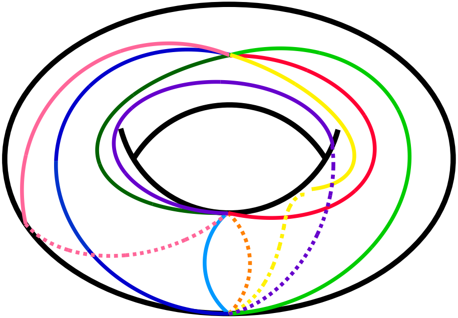
By Menelaus theorem, we have the following dictionary :

- If $col(S)$ is satisfied, then S is a Menelaus configuration;
- If S is a Menelaus configuration and if $col^-(S)$ is satisfied, then all positive colinearities of S are satisfied.

Goal of this work

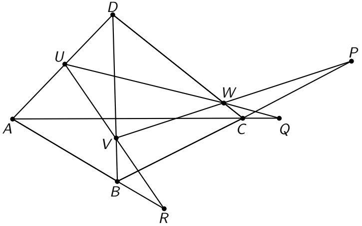
Our intention is to formalise and extend, within proof theory, an idea of Richter-Gebert on incidence theorems, which we paraphrase as follows:

> *If $\mathcal{M}$ is a triangulated manifold that forms a 2-cycle, and therefore is orientable, then the presence of Menelaus configurations on all but one of the triangles automatically implies the existence of a Menelaus configuration on the final triangle.*



Proving incidence theorems: guiding example

Consider the following picture in the Euclidean plane, displaying colinearities (positive and negative):



By the dictionary, we have :

$$\begin{aligned}(C, D; W) \cdot (D, B; V) \cdot (B, C; P) = -1\\ (D, C; W) \cdot (A, D; U) \cdot (C, A; Q) = -1\\ (B, D; V) \cdot (D, A; U) \cdot (A, B; R) = -1,\end{aligned}$$

which, after multiplication and cancellation, delivers

$$(B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1.$$

By the dictionary again, P, Q, R are colinear.

Cyclic sequents

We could have picked any other triple of triangles forming the faces of *ABCD* viewed as a tetrahedron, assuming the corresponding colinearities, and would have derived the colinearity for the missing one.

In logical terms, we have that the following sequent is satisfied:

$$\vdash ABCPQR, ABDVUR, ACDWUQ, BCDWVP$$

where satisfaction means here that whenever 3 out of these 4 sextuples is a Menelaus configuration, then so is the fourth.

This holds more generally in the setting of closed 2-manifolds.

But we can start from any semi-simplical set K and from a 2-cycle on the associated chain complex of $\mathbb{Z}$-modules $C_n = \mathbb{Z}[K_n]$.

Semi-simplicial sets a.k.a. Δ-complexes

An (abstract) Δ-*complex* K consists of mutually disjoint sets $K_0, K_1, \ldots$ and functions $d_i^n \colon K_n \to K_{n-1}$, $n \geq 1$, $0 \leq i \leq n$, which for $l - 1 \geq j$ satisfy

$$d_j^{n-1} \circ d_l^n = d_{l-1}^{n-1} \circ d_j^n.$$

The elements of K_n are the *n-cells* of K, and the functions d_i^n are called *faces*. The intuitive meaning of K_n is that this is the set of (ordered) n-dimensional simplices, and a face d_i^n maps such a simplex to the facet opposite to its ith vertex.

For every $n \geq 0$, let C_n be the free abelian group generated by K_n and let the *boundary* homomorphism $\partial_n \colon C_n \to C_{n-1}$ be defined on generators by

$$\partial_n x = \sum_{i=0}^{n} (-1)^i d_i x.$$

A 2-*cycle* c is an element of C_2 such that $\partial c = 0$.

Homology meets Menelaus: preparations

Let K be a semi-simplicial set, and suppose that we have a 2-cycle c on the associated chain complex (note that there may be none). We can write

$$c = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_{n-1} x_{n-1} + \varepsilon_n x_n$$

were all x_i's are in K_1 and all ε_i's are 1 or -1. There may be repetitions!

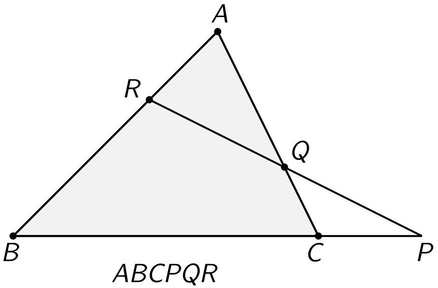
Remark. In this writing, n must be even. This is because writing ∂c yields a sum of $m = 3n$ terms, and because cancellations imply that m is even.

For an arbitrary function $v\colon K_0 \cup K_1 \to \mathbb{R}^2$ (consider the image of a 1-cell as a witness of that cell), consider the following operator $\mu\colon K_2 \to (\mathbb{R}^2)^6$ defined as

$$\mu x = (vd_1 d_2 x, vd_0 d_2 x, vd_0 d_0 x, vd_0 x, vd_1 x, vd_2 x).$$

Homology meets Menelaus: una figura aiuta!

x = superficie grigia



$$\mu x = ABCPQR$$

Homology meets Menelaus: the statement

Proposition. In reference to our chosen cycle

$$c = \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n,$$

for any $1 \leq i \leq n$, if all μx_j 's, for $j \neq i$ are Menelaus configurations, then μx_i is a Menelaus configuration, too.

Idea: c'è un morfismo parziale

$$\text{1-chains (additivo)} \mapsto \mathbb{R}\backslash\{0\} \text{ (multiplicativo)}$$

$$\begin{array}{l} \partial x = BC - AC + AB \\ h(BC) = (B, C; P) \\ h(\partial x) = -1 \quad \text{ssi} \quad ABCPQR \text{ è una configurazione di Menelaus} \end{array}$$

A suitable class of semi-simplicial sets

An $\mathcal{M}$-complex (for Menelaus) is a connected semi-simplicial set L such that:

(0) L has a finite number of cells;

(1) for every $m \geq 3$ the set L_m is empty and every element of $L_0 \cup L_1$ is a face of some element of $L_1 \cup L_2$;

(2) distinct faces map an element of L_{i+1} to distinct elements of L_i $(i \leq 1)$;

(3) every 1-cell of L is a face of exactly two 2-cells of L;

(4) for every $w \in L_0$, the set $L_w = \{u \in L_2 \mid w \text{ is a vertex of } u\}$ is *linked* in the sense that if $u, u' \in L_w$, then there is a sequence of 2-cells starting at u and ending at u', such that every two consecutive 2-cells share an edge having w as a vertex;

(5) L is *orientable*, i.e., the second homology group $H_2(L; \mathbb{Z})$, which consists of all 2-cycles of L, is isomorphic to $\mathbb{Z}$.

No loss of generality for our purposes

Given a semi-simplical set K, a cycle c, and a choice of an expression of c as $\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n$, we can construct

- an $\mathcal{M}$-complex L with n distinct 2-cells $u_1, \ldots, u_n$, with $d = \epsilon_1 u_1 + \ldots + \epsilon_n u_n$ as a generator of $H_2(L; \mathbb{Z})$,
- as well as a morphism $f \colon L \to K$ mapping u_i to x_i.

So, any Euclidean interpretation relative to c can be turned into an interpretation of d. Hence, for the purpose of proving incidence theorems, we do not lose any generality by restricting our attention to $\mathcal{M}$-complexes, which are also a well-behaved class of semi-simplicial sets:

Proposition. The geometric realisation of an $\mathcal{M}$-complex is a closed orientable 2-manifold.

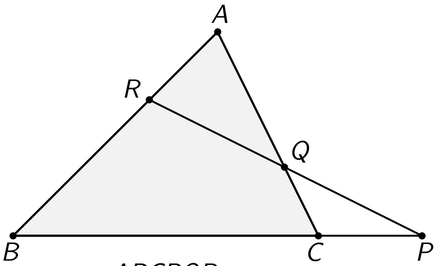
Reformulation of the statement

In the case of $\mathcal{M}$-complexes, the above statement can be reformulated as follows:

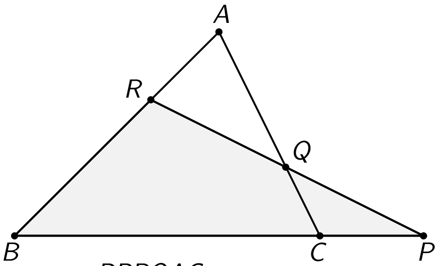
Proposition. If L is an $\mathcal{M}$-complex, with set of 2-faces $\{x_1, \ldots, x_n\}$, then we have: for any $1 \leq i \leq n$, if all μx_j 's, for $j \neq i$ are Menelaus configurations, then μx_i is a Menelaus configuration, too.

Permutation and switching of triangles

- If $A_1A_2A_3B_1B_2B_3$ is a Menelaus configuration and π is a permutation of the set $\{1,2,3\}$, then $A_{\pi(1)}A_{\pi(2)}A_{\pi(3)}B_{\pi(1)}B_{\pi(2)}B_{\pi(3)}$ is a Menelaus configuration.
- If $ABCPQR$ is a Menelaus configuration, then $BPRQAC$, $ARQPCB$ and $CPQRAB$ are Menelaus configurations (as illustrated below).



ABCPQR



BPRQAC

Formal system: formulas and sequents

We fix an arbitrary countable set W. Let

$$F^6(W) = W^6 - \{X_1 \ldots X_6 \in W^6 \mid X_i = X_j \text{ for some } i \neq j\}.$$

The (atomic) formulas of our language are the elements of $F^6(W)$.

In the talk, we limit ourselves to atomic formulas.

We let $\varphi, \psi, \theta, \ldots$ range over formulas.

A *sequent* is a finite multiset Γ of formulas, written $\vdash \Gamma$.

Formal system: the rules

There are two sorts of axioms:

• For every $\mathcal{M}$-complex L such that $L_0 \cup L_1 \subseteq W$, let $\nu\colon L_2 \to F^6(W)$ be defined as $\nu x = (d_1 d_2 x, d_0 d_2 x, d_0 d_0 x, d_0 x, d_1 x, d_2 x)$. Then we set

$$\frac{}{\vdash \{\nu x \mid x \in L_2\}}$$

• The following axioms account for permutations of vertices and switching of triangles.

$$\frac{}{\vdash ABCPQR, BCAQRP} \qquad \frac{}{\vdash ABCPQR, ARQPCB}$$

• In this system, the cut rule looks like this:

$$\frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta}$$

Soundness

Proposition. The Menelaus system is sound.

By this we mean that for every provable sequent $\vdash \Gamma$ and any interpretation (i.e., a sufficiently defined partial function from W to $\mathbb{R}^2$), for all $\phi \in \Gamma$, if the interpretation of each formula in $\Gamma \setminus \{\phi\}$ is a Menelaus configuration, then the interpretation of ϕ is a Menelaus configuration.

Formal system (continued)

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

$$\frac{\vdash \Gamma, \varphi \quad \vdash \Gamma, \psi}{\vdash \Gamma, \varphi \mathbin{\vee\!\!\!\wedge} \psi} \qquad \frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \psi}{\vdash \Gamma, \Delta, \varphi \leftrightarrow \psi}.$$

Comments The first rule is (the cyclic version of) the `mix` rule and the second rule is reminiscent of the &-rule of linear logic.

Soundness (continued)

The soundness extends to connectives, by interpreting the connective $\parr$ as follows:

For every $\varphi \in \Gamma$, every Euclidean interpretation that satisfies every formula in $\Gamma \setminus \{\varphi\}$ also satisfies φ, where every occurrence of $\parr$ in $\Gamma \setminus \{\varphi\}$ is interpreted as disjunction $\vee$ and every occurrence of $\parr$ in φ is interpreted as conjunction $\wedge$. Concerning the connective $\leftrightarrow$, it is always interpreted as classical equivalence.

Decidability

For a multiset Γ of formulae, let $\lambda(\Gamma)$ be the set of elements of W occurring in Γ and let $\kappa(\Gamma)$ be the number of elements (possibly with repetition) of Γ.

Lemma. For every sequent $\vdash \Delta$ that occurs in a derivation of $\vdash \Gamma$, we have that $\lambda(\Delta) \subseteq \lambda(\Gamma)$ and $2 \leq \kappa(\Delta) \leq \kappa(\Gamma)$.

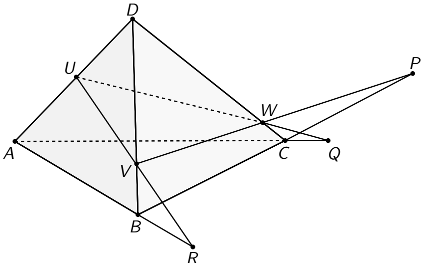
Proposition. The Menelaus system is decidable.

This follows from the lemma (finiteness of the search space), and from the decidability of the properties defining an $\mathcal{M}$-complex.

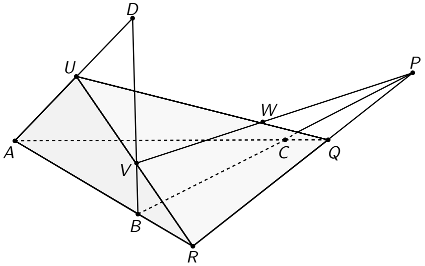
Desargues theorem

Desargue's theorem. If ABC and UVW are two triangles such that $A \neq U$, $B \neq V$ and $C \neq W$, if $BC \cap VW = \{P\}$, $AC \cap UW = \{Q\}$ and $AB \cap UV = \{R\}$, then the lines AU, BV and CW are concurrent if and only if the points P, Q and R are colinear.

Proof of Desargues theorem



$\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR$

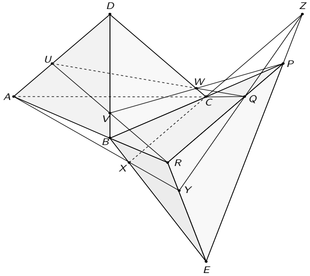


$\vdash ARUVDB, ARQPCB, URQPWV, AQUWDC$

A proof using cuts

Let AU, BV and CW be concurrent lines in $\mathbb{R}^2$, and let X and E be such that B, X and E are colinear. For $\{P\} = BC \cap VW$, $\{Q\} = AC \cap UW$, $\{R\} = AB \cap UV$, $\{Y\} = AX \cap RE$, $\{Z\} = XC \cap EP$, the points Q, Y and Z are colinear. The proof is obtained by cutting three axioms:

$$\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR$$
$$\vdash ABCPQR, BPRQAC$$
$$\vdash BREYXA, BPEZXC, RPEZYQ, BPRQAC$$



Connected sums

The previous example used cuts and the possibilty to switch triangles in a sextuple. If we remove the axioms for permutation and switching of triangles, and stay with the system with cut and only the axioms coming from $\mathcal{M}$-complexes, then cuts are ... useless!

Indeed, let L_1 and L_2 be two $\mathcal{M}$-complexes sharing exactly one 2-cell. Then the result of cutting the two axioms associated with L_1, L_2 is the sequent associated with the connected sum of L_1, L_2 along this 2-cell.

Therefore we can reduce the logical system to allow only non decomposable$\mathcal{M}$-complexes to give rise to axioms.

But this suggests also to study the structure of $\mathcal{M}$-complexes under the opertation of cut / connected sum: they form a cyclic operad, generated by the indecomposable $\mathcal{M}$-complexes. Is this operad fre. The answer is no, and we provide an equational presentation for it.

A mixed proof system: the syntax

For our last example, we need to reason about colinearities, i.e., to be able to derive colinearities from colinearities.

We introduce an additional judgement

$$\Xi \mid\!\cdots \zeta,$$

where Ξ (resp. ζ) is a set of atoms (resp. an atom) of the form ABC or $\neg ABC$ where A, B, C are distinct elements of W (the order of letters does not matter).

We say that an interpretation satisfies ABC (resp. $\neg ABC$) when the points A, B, C are distinct and colinear (resp. not colinear), and that $\Xi \mid\!\cdots \zeta$ is valid if every interpretation that satisfies all atoms of Ξ also satisfies ζ.

Mixed proof system: the rules

We add the following axioms and rules:

$$\frac{\Xi \mathrel{|\!\!\cdots} ABC \qquad \Xi \mathrel{|\!\!\cdots} ABD}{\Xi \mathrel{|\!\!\cdots} BCD}.$$

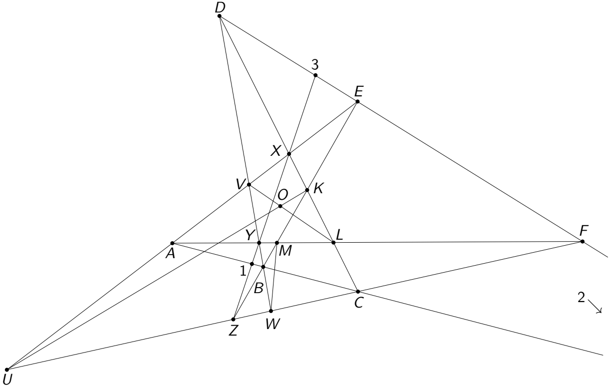
$$\frac{}{\Xi, ABC \mathrel{|\!\!\cdots} ABC}$$

$$\frac{\Xi, A'B'C' \mathrel{|\!\!\cdots} ABC}{\Xi, \neg ABC \mathrel{|\!\!\cdots} \neg A'B'C'}$$

$$\frac{\vdash \Gamma, \phi \quad \Xi \mathrel{|\!\!\cdots} \xi \text{ for all } \psi \in \Gamma \text{ and } \xi \in \mathit{col}(\psi) \quad \Xi \mathrel{|\!\!\cdots} \mathit{col}^{-}(\phi) \quad \zeta \in \mathit{col}^{+}(\phi)}{\Xi \mathrel{|\!\!\cdots} \zeta}$$

Illustrating the mixed system

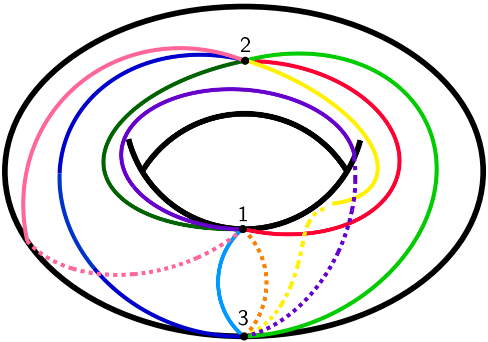
Consider two triples (A, B, C) and (D, E, F) of colinear points, all mutually distinct. Assume that, for $\{X\} = CD \cap AE$ and $\{Z\} = BE \cap CF$, the lines AB, DE and XZ are not concurrent. Let $\{K\} = BE \cap CD$, $\{L\} = AF \cap CD$, $\{M\} = AF \cap BE$, $\{U\} = AE \cap CF$, $\{V\} = AE \cap BD$, $\{W\} = CF \cap BD$. Then the lines KU, LV and MW are concurrent.

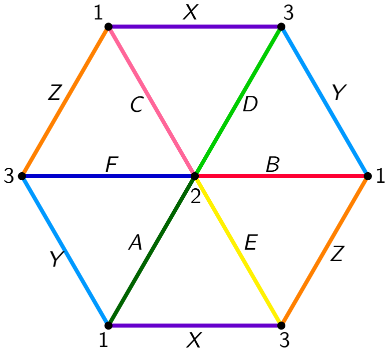


For the proof, we need to name other points (as in the picture in the last slide): $\{1\} = XZ \cap AB$, $\{2\} = AB \cap DE$, $\{3\} = XZ \cap DE$, $\{O\} = KU \cap LV$, and $\{Y\} = 13 \cap BD$.

We use an axiom coming from a triangulation of the torus:

$$\text{Axiom 1} \quad \vdash 123EXA, 123EZB, 123DYB, 123DXC, 123FZC, 123FYA.$$





We also use the axiom for the tetrahedron $UXZK$:

$$\text{Axiom 2} \quad \vdash UXKLOV, UXZYWV, KXZYML, UZKMOW.$$

And in the middle we use reasoning on colinearity, resulting in the following proof architecture:

$$\dfrac{\dfrac{\dfrac{}{\Xi \mid\!\cdots MAF} \quad \dfrac{}{\Xi \mid\!\cdots FYA}\;(\text{Axiom 1}) \quad \dfrac{}{\Xi \mid\!\cdots LAF}}{\Xi \mid\!\cdots YML}}{\Xi \mid\!\cdots WMO}\;(\text{Axiom 2})$$

Where the rule in the middle deriving YML is an easily derived rule about colinearities.

Connected sums: the definition

Let L_1, L_2 be two $\mathcal{M}$-complexes, let x (resp. y) be a 2-cell of L_1 (resp. L_2), suppose that $(L_1)_2 \setminus \{x\} \cap (L_2)_2 \setminus \{y\} = \emptyset$, and that $(L_1)_i \cap (L_2)_i = \emptyset$ for $i < 2$. Then the connected sum of L_1, L_2 is defined as the $\mathcal{M}$-complex obtained by removing x, y and by glueing their faces.

Proposition $\mathcal{M}$-complexes are stable under connected sums.

This gives rise to a cylic operad. We identify $\mathcal{M}$-complexes up to < 2-isomorphims, i.e., isomorphisms of the form (ϕ_0, ϕ_1, id).

For a given set X, we set $\mathcal{C}(X)$ to be the set of < 2-isomorphism classes of $\mathcal{M}$-complexes having X as set of 2-cells.

This gives rise to a collection (or species), by defining the action of a bijection $\sigma : Y \to X$ as follows. We define a representative K' of $[K]^\sigma$ as follows: $K'_2 = Y$, $K'_i = K_i$ for $i < 2$, and the face maps of K' coincide with those of K except for the maps d_i^2 which are defined by $d_i^2 u = d_i^2(\sigma(u))$.

The Menelaus cyclic operad

- Compositions are given by connected sums.
- Identities are given by the $\mathcal{M}$-complexes with exactly two 2-cells sharing their 3 faces.

This cyclic operad is called the Menelaus cyclic operad.

Our goal in the rest of the talk (quite independent from the considerations of the first part) is to give a presentation of the Menelaus cyclic operad by generators and relations.

We shall first characterise $\mathcal{M}$-complexes that cannot be obtained as connected sums of smaller $\mathcal{M}$-complexes.

Let K be an $\mathcal{M}$-complex and let $T = \{e_0, e_1, e_2\} \subseteq K_1$ be such that $\partial_1(e_0 - e_1 + e_2) = 0$, i.e. $e_0 - e_1 + e_2$ is a 1-cycle. Consider the binary relation on K_2 of sharing an edge from $K_1 - T$. Let τ be the transitive closure of this relation. We say that T is a *cut-triangle*, when τ is an equivalence relation with exactly two classes. If K contains a cut-triangle, then we say that it is *reducible*, otherwise it is *irreducible*.

Proposition. An $\mathcal{M}$-complex K is reducible if and only if it can be obtained as a connected sum of two simpler (with respect to the cardinality of the set of 2-cells) $\mathcal{M}$-complexes.

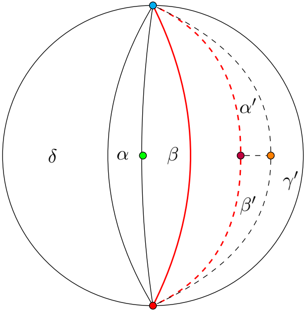
A key technical lemma

Recall the notation $L_w = \{u \in L_2 \mid w \text{ is a vertex of } u\}$, for a 0-cell w. We say that two 2-cells are w-neigbours if they share a face having w as a vertex.

Camembert lemma. The axiom (4) in the definition of $\mathcal{M}$-complex can be reinforced as folows. For every 0-cell w, the 2-cells of the link L_w at w can be displayed without repetition around w in a circle. Formally, we can arrange a cyclic order on $L_w = \{u_1, \ldots, u_n\}$ in such a way that u_i and u_{i+1} are w-neighbours, modulo n.

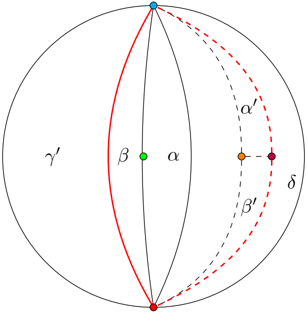
The Menelaus cyclic operad is not free

We take as generators the irreducible $\mathcal{M}$-complexes.



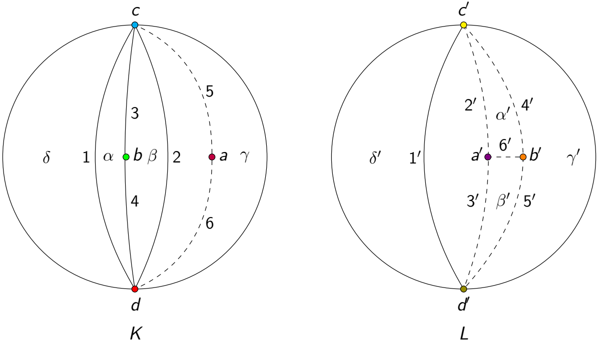
$[K\,{}_{\gamma}\circ_{\delta'}\,L]$

$=$



$[K^{\sigma}\,{}_{\varphi}\circ_{\psi}\,L^{\tau}]$

where



and where

- σ renames the 2-cells α, β, γ and δ of *K* into β, α, γ' and φ, respectively,
- τ renames the 2-cells α', β', γ' and δ' of *L* into α', β', δ and ψ, respectively.

Imbricated cut-triangles

Let K be an $\mathcal{M}$-complex and let $T_1 \subseteq K_1$ and $T_2 \subseteq K_1$ be two cut-triangles of K. We say that T_1 is *disjoint* from T_2 if all the edges of T_1 are 1-cells of one of the two $\mathcal{M}$-complexes induced by T_2.

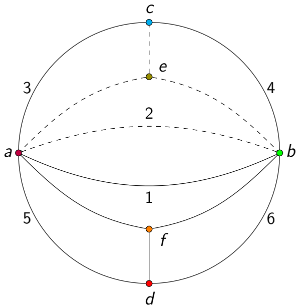
Lemma. The relation of disjointness is symmetric.

Two cut-triangles of the same $\mathcal{M}$-complex that are not disjoint are called *imbricated*.

A sufficient condition for disjointness. Let T_2 be a cut-triangle. If T_1 is a cut-triangle distinct from T_2 such that whenever two vertices A, B of T_1 are vertices of T_2, then also the edge of T_1 connecting A, B is an edge of T_2, then T_1 is disjoint from T_2.

This condition is not necessary

The following $\mathcal{M}$-complex contains disjoint cut-triangles, $T_1 = \{2, 3, 4\}$ and $T_2 = \{1, 5, 6\}$, that share two vertices (and no edge):



A (wrong) promise of freeness

Lemma. For two disjoint cut-triangles $T_1 \subseteq K_1$ and $T_2 \subseteq K_1$ of an $\mathcal{M}$-complex K, there exist $\mathcal{M}$-complexes K_1, K_2 and K_3 such that

$$K = (K_1 \, {}_{t_1}\!\circ_{t_1} K_2) \, {}_{t_2}\!\circ_{t_2} K_3 = K_1 \, {}_{t_1}\!\circ_{t_1} (K_2 \, {}_{t_2}\!\circ_{t_2} K_3),$$

where t_1 and t_2 are the 2-cells induced by T_1 and T_2.

A presentation of the Menelaus cyclic operad

Theorem. The Menelaus cyclic operad is the quotient of the free cyclic operad generated by the irreducible $\mathcal{M}$-complexes, under the equivalence relation generated by the equalities of the form

$$\mathcal{T}_1 \,{}_u\!\circ_v \mathcal{T}_2 = \mathcal{T}_1' \,{}_{u'}\!\circ_{v'} \mathcal{T}_2',$$

for each quadruple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_1', \mathcal{T}_2')$ of unrooted trees and quadruple $(u, v, u' v')$ of leaves, such that both hand sides are well formed and evaluate, up to $<$2-isomorphism, to the same $\mathcal{M}$-complex K, in which the cut-triangles T and T', associated with the pairs (u, v) and (u', v'), are imbricated.